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On spacelike and timelike minimal surfaces in AdS_n

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Abstract: We discuss timelike and spacelike minimal surfaces in AdS_n using a Pohlmeyer type reduction. The differential equations for the reduced system are derived in a parallel treatment of both type of surfaces, with emphasis on their characteristic differences. In the timelike case we find a formulation corresponding to a complete gauge fixing of the torsion. In the spacelike case we derive three sets of equations, related to different parameterizations enforced by the Lorentzian signature of the metric in normal space. On the basis of these equations, we prove that there are no flat spacelike minimal surfaces in AdS_n, $n \geq 4$ beyond the four cusp surfaces used in the Alday-Maldacena conjecture. Furthermore, we give a parameterization of flat timelike minimal surfaces in AdS_5 in terms of two chiral fields.

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Contents

1 Introduction

According to a remarkable conjecture put forward by Alday and Maldacena [1], the p-point gluon scattering amplitude at strong coupling in $\mathcal{N} = 4$ super Yang-Mills is related to a string worldsheet in AdS_5 approaching a p-sided polygon spanned by the lightlike momenta of the scattering process on the conformal boundary of AdS_5. In ref. [1] this conjecture has been checked for $p = 4$. Furthermore, taking it for granted, the breakdown of the BDS [3] ansatz for gluon amplitudes has been anticipated by estimating the behaviour of the string world surface for large p [2]. To fully establish the conjectured amplitude-string correspondence one needs to solve the generalized Plateau problem for lightlike polygonal boundaries. Since the related mathematical literature is mostly devoted to spaces with positive definite metric, one is faced with a deep and delicate problem, and despite a lot of effort [4–7] so far no real breakthrough has been achieved beyond $p = 4$.

The worldsheets constructed in [1] for $p = 4$ and generic kinematics of the gluon momenta are all $\mathrm{SO}(2,4)$ transforms of a highly symmetric configuration embedded in an $AdS_3 \subset AdS_5$. For this AdS_3 solution the worldsheet approaches a lightlike tetragon winding alternating up and down around the conformal boundary of AdS_3, the cylinder $\mathbb{R} \times S^1$, with each side just extending in a quarter of the cylinder. By construction the surface is minimal. On top of this, by direct inspection, one finds that the surface is flat, too.

Given the high symmetry of this AdS_3 solution it is naturally to ask, whether one could find solutions in the subset of flat minimal surfaces also for e.g. AdS_4 and a hexagon winding in a maximal symmetric way around $\mathbb{R} \times S^2$ or for AdS_5 and an octagon winding around

$\mathbb{R} \times S^3$. Furthermore, the surface of ref. [1] is spacelike. Although we are not aware of a rigorous proof that all solutions with lightlike closed polygonal boundaries winding around the conformal boundary of AdS_5 are spacelike, we expect this to be valid. For this reason, in respect to the Alday-Maldacena conjecture, we concentrate on spacelike minimal surfaces.

But in parallel a look on timelike minimal surfaces is in order. They describe the dynamics of strings in real time. As emphasized in ref. [4] the solution of ref. [1] can be obtained from a rigid open string rotating in a plane (in its limit of infinite extension) by Wick rotation of both the worldsheet time and some target space coordinates. There are also dynamical rigid string solutions in AdS_5 describing a string performing two independent rotations in the (X^1, X^2) and the (X^3, X^4)-plane which are flat [8]. In this case a Wick rotation of these solutions does not bring us back in an AdS_5. But nevertheless, it seems to be open, whether similar to the timelike case, there exist flat minimal surfaces also in the spacelike case, beyond the known tetragon solution of ref. [1], which wind in the full AdS_5 and cannot be embedded in an AdS_3 trivially extended to AdS_5.

By classical theorems of differential geometry the embedding of surfaces in higher dimensional manifolds is controlled by the system of Gauß, Codazzi-Mainardi and Ricci equations. If these equations are fulfilled, the surface is fixed up to isometries in the embedding space. An early discussion of strings in AdS_4 along these lines has been given in ref. [9].

In the present paper we follow an equivalent procedure developed originally for the reduction of the dynamics of the $O(N)$ sigma model [10] and applied to the dynamics of strings in de Sitter and anti de Sitter spaces in [4, 11, 12]. Our main focus will be on the parallel treatment for both timelike (i.e. dynamical) and spacelike minimal surfaces and the discussion of their characteristic differences. Based on this, we can prove that there are no flat minimal spacelike surfaces in AdS_n beyond those constructed in [1], and can parameterize all flat timelike surfaces in AdS_5 by two free chiral fields. We also comment on the reduction for arbitrary dimensions AdS_n.

2 The general framework for minimal surfaces in AdS_n

Minimal surfaces with coordinates $z^\mu = (\sigma, \tau)$ embedded in a space parameterized by coordinates X^k are solutions of the equation

$$g^{\mu\nu}\left(\nabla_\mu \partial_\nu X^k(z) + \partial_\mu X^j \partial_\nu X^l \; \Gamma^k_{jl}(X(z))\right) = 0\,, \tag{2.1}$$

with Γ^k_{jl} denoting the Christoffel symbols in the embedding space, $g_{\mu\nu}$ the induced metric and ∇_μ the induced two-dimensional covariant derivative. This guarantees the vanishing of all mean curvatures, and it is also the stationarity condition for the two-dimensional volume functional (Nambu-Goto action). Realizing AdS_n as a hyperboloid in $\mathbb{R}^{2,n-1}$

$$(Y^0(X))^2 + (Y^{0'})^2 - (Y^1)^2 - \cdots - (Y^{n-1})^2 = 1 \tag{2.2}$$

and choosing conformal coordinates on the surface one gets from (2.1)

$$\partial\bar{\partial} Y^N(X(z)) - \partial Y^K \bar{\partial} Y_K \; Y^N = 0\ . \tag{2.3}$$

The choice of conformal coordinates gives the additional condition

$$\partial Y^N \partial Y_N = \bar\partial Y^N \bar\partial Y_N = 0\,, \tag{2.4}$$

where ∂, $\bar\partial$ are defined by $\partial = \partial_\sigma + \partial_\tau$, $\bar\partial = \partial_\sigma - \partial_\tau$ for timelike surfaces and by $\partial = \partial_\sigma - i\partial_\tau$, $\bar\partial = \partial_\sigma + i\partial_\tau$ for spacelike surfaces.

One now extends the vectors $Y, \partial Y, \bar\partial Y$ to a basis of $\mathbb{R}^{2,n-1}$ [4, 11]

$$\{e_N\} = \{Y, \partial Y, \bar\partial Y, B_4, \ldots, B_{n+1}\}\ . \tag{2.5}$$

The orthonormal vectors B_a pointwise span the normal space of the surface inside AdS_n. By eq. (2.2) Y is timelike. For timelike surfaces a further timelike vector is parallel to the surface, hence the normal space has to be positive definite. In contrast for spacelike surfaces the second timelike vector has to be in the normal space. With $(a, b = 4, \ldots, n+1)$

$$h_{ab} = \delta_{ab} \text{ or } \eta_{ab}, \quad \text{for timelike or spacelike surface,} \tag{2.6}$$

we require

$$(B_a, B_b) = h_{ab}\,, \qquad (B_a, Y) = (B_a, \partial Y) = (B_a, \bar\partial Y) = 0\ . \tag{2.7}$$

Moving the basis (2.5) along the surface one gets

$$\partial\ e_N = A_N{}^K\ e_K\,, \qquad \bar\partial\ e_N = \bar A_N{}^K\ e_K\ . \tag{2.8}$$

Now the strategy is to find a suitable parameterization of the dynamical (geometrical) degrees of freedom in the entries of the matrices A, $\bar A$ and to derive differential equations for the corresponding functions, using the equation of motion (minimal surface condition) (2.3) and the integrability condition for eq. (2.8). Then, after solving these differential equations, the surface has to be reconstructed by integration of (2.8).

Introducing

$$\alpha(\sigma,\tau) = \log(\partial Y, \bar\partial Y) \tag{2.9}$$

$$\begin{aligned} u_a(\sigma,\tau) &= (B_a, \partial\partial Y)\,, & \bar u_a(\sigma,\tau) &= (B_a, \bar\partial\bar\partial Y)\,, \\ A_{ab} &= (\partial B_a, B_b)\,, & \bar A_{ab} &= (\bar\partial B_a, B_b)\,, \end{aligned} \tag{2.10}$$

and using (2.3), (2.7) one can give eqs. (2.8) a more detailed form

$$\begin{aligned} \partial Y &= \partial Y \\ \partial\partial Y &= \partial\alpha\partial Y \qquad\qquad + u^b B_b \\ \partial\bar\partial Y &= e^\alpha Y \\ \partial B_a &= \qquad\quad -e^\alpha\ u_a \bar\partial Y\ + A_a{}^b B_b\,, \end{aligned} \tag{2.11}$$

as well as the equations which one gets by the replacements $\partial \leftrightarrow \bar\partial$, $u_a \to \bar u_a$, $A_a{}^b \to \bar A_a{}^b$.[1] Indices on $u, \bar u$ and $A, \bar A$ are raised and lowered with the normal space metric h, see eq. (2.6). A and $\bar A$ with both indices downstairs are antisymmetric.

[1]Note that for timelike surfaces u and $\bar u$ as well as A and $\bar A$ are real. On the other side, for spacelike surfaces u and A are complex, and then the bar means complex conjugation.

Then, the integrability condition $\partial\bar{\partial}e_N = \bar{\partial}\partial e_N$ for eq. (2.8) gives

$$\partial\bar{\partial}\alpha - e^{-\alpha}u^b\bar{u}_b - e^{\alpha} = 0\,, \tag{2.12}$$

$$\partial\bar{u}_a - \ A_a{}^b\bar{u}_b = 0\,, \qquad\qquad \bar{\partial}u_a - \ \bar{A}_a{}^b u_b = 0\,, \tag{2.13}$$

$$e^{-\alpha}\left(\bar{u}_a u^b - u_a\bar{u}^b\right) = \partial\bar{A}_a{}^b - \bar{\partial}A_a{}^b + \bar{A}_a{}^c A_c{}^b - A_a{}^c\bar{A}_c{}^b\,. \tag{2.14}$$

Here, a comment on the geometrical meaning of our quantities α, u, A is in order. Since we are using conformal coordinates,

$$R = -2\,e^{-\alpha}\,\partial\bar{\partial}\alpha \tag{2.15}$$

is the curvature scalar on our surface. u, $\bar{u}$ parameterize the second fundamental forms $l^c_{\mu\nu} = (B^c, \partial_\mu\partial_\nu Y)$ with built in minimal surface condition $l^{c\ \mu}_{\ \mu} = 0$. Writing for timelike surfaces $u = a + b$ and $\bar{u} = a - b$ one gets

$$l^c_{11} = l^c_{22} = \frac{1}{2}\,a^c\,, \qquad\qquad l^c_{12} = l^c_{21} = \frac{1}{2}\,b^c\,, \tag{2.16}$$

and for spacelike surfaces with $u = a + ib$, $\bar{u} = a - ib$

$$l^c_{11} = -l^c_{22} = \frac{1}{2}\,a^c\,, \qquad\qquad l^c_{12} = l^c_{21} = -\frac{1}{2}\,b^c\,. \tag{2.17}$$

The matrices A, $\bar{A}$ in (2.13), (2.14) describe the torsion of the surface (for AdS_n, $n \geq 4$). Eqs. (2.12)–(2.14) are the Gauß, Codazzi-Mainardi and Ricci equations specialized to minimal surfaces in conformal coordinates. Eq. (2.12) can be related to the Gauß equation in two ways. One version concerns the relation between the difference of the scalar curvature of the surface and the constant curvature of AdS to the second fundamental forms (with zero mean curvature) in the normal space in AdS only. The other version concerns the embedding in $\mathbb{R}^{2,n-1}$, now the big space is flat, and one has one more second form, whose mean curvature is of course not zero.

The further analysis depends crucially on the signature of the induced metric on the surface.

3 Timelike minimal surfaces in AdS_n

In this case all quantities in (2.12)–(2.14) are real and the metric in the normal space is positive definite, see (2.6). ∂ and $\bar{\partial}$ are the derivatives with respect to the chiral coordinates $z = \frac{1}{2}(\sigma + \tau)$, $\bar{z} = \frac{1}{2}(\sigma - \tau)$. Due to the antisymmetry of A and $\bar{A}$ one gets from eq. (2.13)

$$\bar{\partial}(u^a u_a) = 0\,, \qquad\qquad \partial(\bar{u}^a\bar{u}_a) = 0\,. \tag{3.1}$$

Under a conformal transformation $z \mapsto \zeta(z)$, $\bar{z} \mapsto \bar{\zeta}(\bar{z})$ the definitions (2.10) imply:
$u \mapsto (\zeta')^{-2}u$, $\ \bar{u} \mapsto (\bar{\zeta}')^{-2}\bar{u}$. This can be used to achieve within the conformal gauge

$$u^a u_a = 1 = \bar{u}^a\bar{u}_a\,. \tag{3.2}$$

There are exceptional cases, if either both or one out of $u^a u_a$ and $\bar{u}^a \bar{u}_a$ are zero. If both are zero, due to the positive definiteness, u and $\bar{u}$ are zero, which implies the vanishing of all second fundamental forms (with respect to AdS_n). The surface is then (part of) an $AdS_2 \subset AdS_n$. The exceptional case $u^a u_a = 1$ and $\bar{u}^a \bar{u}_a = 0$ will be postponed to the end of this section.

For a given surface, the choice of the normal vectors B_a in (2.7) is fixed only up to a $(z, \bar{z})$-dependent SO$(n-2)$ transformation, which effects $u, \bar{u}$ and $A, \bar{A}$ as

$$\begin{aligned} u_a &\mapsto \Omega_a{}^b\, u_b\,, & \bar{u}_a &\mapsto \Omega_a{}^b\, \bar{u}_b\,, \\ A_a{}^b &\mapsto \left(\Omega A \Omega^{-1} + \partial\Omega\ \Omega^{-1}\right)_a{}^b\,, & \bar{A}_a{}^b &\mapsto \left(\Omega \bar{A} \Omega^{-1} + \bar{\partial}\Omega\ \Omega^{-1}\right)_a{}^b\,. \end{aligned} \tag{3.3}$$

We now want to use this gauge freedom to simplify eqs. (2.12)–(2.14). Starting with light cone gauge $\bar{A} = 0$, we get from (2.13) $\bar{\partial} u = 0$. Then, with a gauge transformation depending only on z, we can bring u_a to the form $u_a = \delta_{a,n+1}$. There is no possibility to simplify $\bar{u}$, beyond making use of (3.2), and we continue with

$$u_a = (0, 0, \ldots, 1)\,, \qquad \bar{u}_a = (\chi_4, \chi_5, \ldots, \chi_n, \pm\sqrt{1 - \chi\cdot\chi}\ )\,. \tag{3.4}$$

Inserting all this into eq. (2.14), we see that the field strength on the r.h.s. no longer contains the commutator term and is given by $-\bar{\partial} A$. Furthermore, due to the structure of the l.h.s. and the special form of u, $\bar{u}$ all its matrix elements are zero, except those in the last row and column. Then in addition, with a z dependent gauge transformation, acting only in the space orthogonal to B_{n+1}, we can also achieve zeros for all matrix elements of A, except those in the last row or column[2]

$$A_a{}^b = \begin{pmatrix} 0 & \cdots & 0 & \lambda_4 \\ 0 & \cdots & 0 & \lambda_5 \\ & & & \cdot \\ & & & \cdot \\ 0 & \cdots & 0 & \lambda_n \\ -\lambda_4 & \cdots & -\lambda_n & 0 \end{pmatrix}\,, \qquad \bar{A}_a{}^b = 0\,. \tag{3.5}$$

Inserting this parameterization into (2.13) and (2.14) one finds

$$\lambda_a = \pm\frac{\partial\chi_a}{\sqrt{1 - \chi\cdot\chi}}\,, \qquad \bar{\partial}\lambda_a = -e^{-\alpha}\chi_a\,. \tag{3.6}$$

After this complete gauge fixing we arrive at a nonlinear coupled system of second order differential equations for the $(n-2)$ functions $\alpha,\ \chi_4, \ldots, \chi_n$

$$\partial\bar{\partial}\alpha\ \mp\ \sqrt{1 - \chi\cdot\chi}\ \ e^{-\alpha}\ - e^{\alpha} = 0\,, \tag{3.7}$$

$$\partial\bar{\partial}\chi_b\ \pm\ \sqrt{1 - \chi\cdot\chi}\ \ e^{-\alpha}\ \chi_b + \frac{\chi\cdot\bar{\partial}\chi}{1 - \chi\cdot\chi}\ \partial\chi_b = 0\,. \tag{3.8}$$

These equations have a similar structure to those derived for the $O(N)$ sigma model in [13].

[2] At this point our analysis is restricted to simple connected patches. On the global level putting these A elements to zero could be obstructed by nonzero holonomies along some cycles.

For AdS_3 there are no χ_a, and one ends with one equation for α: $\partial\bar{\partial}\alpha - 2\cosh\alpha = 0$ or $\partial\bar{\partial}\alpha - 2\sinh\alpha = 0$, depending on whether the signs of u_4 and $\bar{u}_4$ are equal or opposite. In refs. [4, 11] only the sinh version is discussed.

For AdS_4 besides α, there is only χ_4. With the parameterization $\pm\sqrt{1-\chi_4^2} = \cos\beta$ one gets [11]

$$\begin{aligned} \partial\bar{\partial}\alpha - e^{-\alpha}\cos\beta - e^{\alpha} &= 0 \\ \partial\bar{\partial}\beta + e^{-\alpha}\sin\beta &= 0\,. \end{aligned} \tag{3.9}$$

We still have to comment the one exceptional case $u^a u_a = 1, \quad \bar{u}^a\bar{u}_a = 0$, postponed above. Repeating the arguments of the generic case, but with all $\bar{u}_a = 0$, one further gets $\partial\bar{\partial}\alpha - e^{\alpha} = 0, \quad u_a = \delta_{a,n+1}$ and all ${A_a}^b,\ {\bar{A}_a}^{\ b}$ zero. This gives a constant curvature surface isometric to AdS_2. But since one of the second fundamental forms is not identically zero, the embedding in $AdS_n,\ n > 2$ is not totally geodesic.

4 Spacelike minimal surfaces in AdS_5

Now ∂ and $\bar{\partial}$ are the derivatives with respect to the surface complex coordinates $z = \frac{1}{2}(\sigma + i\tau),\ \bar{z} = \frac{1}{2}(\sigma - i\tau)$ and the bar on u^c and ${A_b}^c$ implies complex conjugation, too. Eq. (3.1) holds as in the timelike case, and by a conformal (holomorphic) transformation $z \mapsto \zeta(z),\ \bar{z} \mapsto \overline{\zeta(z)}$ one can achieve eq. (3.2) (the exceptional case $u^a u_a = 0$ we discuss later). With $u_c = a_c + ib_c$ this means

$$a^c\, a_c\ - b^c\, b_c = 1\,, \qquad\qquad a^c\, b_c = 0\,. \tag{4.1}$$

The sign of $a^c\, a_c$ and $b^c\, b_c$ is indefinite. However, in a space with just one timelike direction, see (2.6), the second equation in (4.1) forbids that both of these terms are negative. Therefore we end up with three cases: $b^c\, b_c\ >\ 0;\ \ -1\ \leq\ b^c\, b_c\ <\ 0;\ \ b^c\, b_c\ = 0\,.$

Unfortunately, we did not find yet a simple completely gauge fixed formulation similar to the previous section for generic AdS_n. For this reason we now consider AdS_5, which after all is our main focus.

Making use of the gauge freedom (3.3), but now with $\Omega \in O(1,2)$, one can give u^c the following form (taking B_4 as the timelike vector in the normal space and β real)

$$\text{spacelike I} \qquad (b^c\, b_c\ >\ 0)\,, \qquad u^c = \left(0, i\sinh\frac{\beta}{2}, \cosh\frac{\beta}{2}\right) \tag{4.2}$$

$$\text{spacelike II} \qquad (-1\ \leq\ b^c\, b_c\ <\ 0)\,, \qquad u^c = \left(i\sin\frac{\beta}{2}, \cos\frac{\beta}{2}, 0\right) \tag{4.3}$$

$$\text{spacelike III} \qquad (b^c\, b_c\ = 0)\,, \qquad u^c = (1 + i\beta, 1 + i\beta, 1)\,. \tag{4.4}$$

We now discuss case spacelike I in some detail. As input in the Gauß equation (2.12) one gets $u^c\bar{u}_c = \cosh\beta$. Inserting the u-parameterization (4.2) into (2.13) one finds ${A_5}^6 = -\frac{i}{2}\partial\beta$ and the condition $i{A_4}^5\sinh\frac{\beta}{2} = {A_4}^6\cosh\frac{\beta}{2}$, which leads to the parameterization ${A_4}^5 = \rho\cosh\frac{\beta}{2}\,, \quad {A_4}^6 = i\rho\sinh\frac{\beta}{2}$. Eq. (2.14) then gives three more differential equations for $\beta,\ \rho,\ \bar{\rho}$, and altogether we end up with

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case spacelike I (u^c from (4.2)):

$$A_5{}^6 = -\frac{i}{2}\,\partial\beta\,, \qquad A_4{}^5 = \rho\cosh\frac{\beta}{2}\,, \qquad A_4{}^6 = i\rho\sinh\frac{\beta}{2}\,. \tag{4.5}$$

$$\partial\bar\partial\alpha - e^{-\alpha}\cosh\beta - e^{\alpha} = 0\,, \tag{4.6}$$

$$\partial\bar\partial\beta + (e^{-\alpha} + \rho\bar\rho)\ \sinh\beta = 0\,, \tag{4.7}$$

$$(\bar\rho\partial\beta - \rho\bar\partial\beta)\ \sinh\frac{\beta}{2} + (\partial\bar\rho - \bar\partial\rho)\ \cosh\frac{\beta}{2} = 0\,, \tag{4.8}$$

$$(\bar\rho\partial\beta + \rho\bar\partial\beta)\ \cosh\frac{\beta}{2} + (\partial\bar\rho + \bar\partial\rho)\ \sinh\frac{\beta}{2} = 0\ . \tag{4.9}$$

Similarly one gets for *case spacelike II* (u^c from (4.3)):

$$A_4{}^5 = \frac{i}{2}\,\partial\beta\,, \qquad A_4{}^6 = \rho\cos\frac{\beta}{2}\,, \qquad A_5{}^6 = i\rho\sin\frac{\beta}{2}\,. \tag{4.10}$$

$$\partial\bar\partial\alpha - e^{-\alpha}\cos\beta - e^{\alpha} = 0\,, \tag{4.11}$$

$$\partial\bar\partial\beta + (e^{-\alpha} + \rho\bar\rho)\ \sin\beta = 0\,, \tag{4.12}$$

$$(\bar\rho\partial\beta - \rho\bar\partial\beta)\ \sin\frac{\beta}{2} - (\partial\bar\rho - \bar\partial\rho)\ \cos\frac{\beta}{2} = 0\,, \tag{4.13}$$

$$(\bar\rho\partial\beta + \rho\bar\partial\beta)\ \cos\frac{\beta}{2} + (\partial\bar\rho + \bar\partial\rho)\ \sin\frac{\beta}{2} = 0\ . \tag{4.14}$$

Note that the differential equations for case II are related to those of case I by $\beta \mapsto i\beta$.

To be complete, we also give

case spacelike III (u^c from (4.4)):

$$A_4{}^5 = \rho\,, \qquad A_4{}^6 = -A_5{}^6 = i\partial\beta - \rho(1 - i\beta)\,, \tag{4.15}$$

$$\partial\bar\partial\alpha - 2\cosh\alpha = 0\,, \tag{4.16}$$

$$\partial\bar\partial\beta + (e^{-\alpha} + \rho\bar\rho)\beta + (\bar\rho\partial\beta + \rho\bar\partial\beta) + (\partial\bar\rho + \bar\partial\rho)\ \frac{\beta}{2} = 0\,, \tag{4.17}$$

$$\partial\bar\rho - \bar\partial\rho = 0\ . \tag{4.18}$$

Let us add some comments. In the formulation, given in the previous section for timelike surfaces in AdS_5, we needed three real valued functions α, χ_4, χ_5, obeying a system of second order differential equations. Here we have real α, β and one complex ρ, but since the differential equations for ρ, $\bar\rho$ are of first order only, the overall counting of degrees of freedom matches.

There is of course also a description of timelike minimal surfaces in AdS_5, in parallel to the treatment of this section. The resulting differential equations coincide with those for case spacelike II up to one difference: in eqs. (4.13), (4.14) ρ has to be replaced by $-\rho$. But the crucial point is that per se $\rho\bar\rho$ can have both signs, while it is positive semidefinite for spacelike surfaces. This will have far reaching consequences for the existence of flat minimal surfaces, as will be discussed in the next sections.

For AdS_3 there is only one u, namely u^4 and no ρ. Then eq. (4.1) means $-a_4a_4 + b_4b_4 = 1$ and $a_4b_4 = 0$. This necessarily implies $a_4 = 0$ and $b_4 = \pm 1$, hence $u_4\bar u^4 = -1$, and one is left with the sinh-Gordon equation for α. In contrast to the timelike case, here the cosh variant is excluded.

In AdS_4 there is not enough freedom to realize cases spacelike I or spacelike III, one also has not to introduce ρ. The equations for α and β then have the same form (3.9) as in the timelike case.

We close with the discussion of the postponed exceptional case $u^a u_a = 0$. Instead of (4.1) one has $a^c\, a_c - b^c\, b_c = 0,\;\; a^c\, b_c = 0$, implying $b^c b_c \geq 0$. To avoid a treatment in all details, let us concentrate on the issues relevant for the search for flat surfaces in the next section. The case $b^c b_c = 0$ gives $u^a \bar{u}_a = 0$ and via (2.12) and (2.15) a spacelike surface of constant negative scalar curvature, i.e. $\mathbb{H}^2$. The case $b^c b_c > 0$ allows a parameterization $u^a \bar{u}_a = e^{\beta}$. This leads to the absence of flat solutions of (2.12) within the exceptional cases.

5 Flat spacelike minimal surfaces

On a flat surface one can always choose coordinates in which the induced metric is $\eta_{\mu\nu}$ or $\delta_{\mu\nu}$, respectively. However, we have already completely used up the freedom of coordinate transformations by first starting with conformal coordinates and then using the remaining conformal transformations to get (3.2). Therefore, for flat surfaces we have to allow also non constant α with $\partial\bar{\partial}\alpha = 0$, see eq. (2.15).

Let us start with AdS_3. Then from the sinh-Gordon equation one necessarily gets $\alpha = 0$. The matrices $A_N^{\;K}$ and $\bar{A}_N^{\;K}$ that have to be used in the surface reconstruction equation (2.8) are (the timelike case has been discussed in [4], where all entries were real)

$$A_N^{\;K} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix}, \qquad \bar{A}_N^{\;K} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{5.1}$$

Above we had as an alternative $u_4 = \pm i$, we take here $u_4 = i$. The other choice can be generated by $B_4 \mapsto -B_4$ or $\tau \mapsto -\tau$ and describes a surface related by a sign reversal of one of the embedding coordinates in $\mathbb{R}^{2,2}$.

The solution of (2.8) is now

$$e_N(\sigma,\tau) = \mathcal{M}_N^{\;K}\; e_K(0,0)\,, \qquad \mathcal{M}_N^{\;K} = \left(\exp\left(\frac{\sigma+i\tau}{2}\;A\right)\;\exp\left(\frac{\sigma-i\tau}{2}\;\bar{A}\right)\right)_N^{\;\;K}. \tag{5.2}$$

The explicit exponentiation yields

$$\mathcal{M}_N^{\;K} = \begin{pmatrix} C_\sigma C_\tau & i\,\bar{U}_{\sigma,\tau} & -i\,U_{\sigma,\tau} & S_\sigma S_\tau \\ -i\,U_{\sigma,\tau} & C_\sigma C_\tau & -i\,S_\sigma S_\tau & \bar{U}_{\sigma,\tau} \\ i\,\bar{U}_{\sigma,\tau} & i\,S_\sigma S_\tau & C_\sigma C_\tau & U_{\sigma,\tau} \\ S_\sigma S_\tau & U_{\sigma,\tau} & \bar{U}_{\sigma,\tau} & C_\sigma C_\tau \end{pmatrix}, \tag{5.3}$$

with

$$C_\sigma = \cosh\frac{\sigma}{\sqrt{2}}\,, \quad S_\sigma = \sinh\frac{\sigma}{\sqrt{2}}\,, \quad U_{\sigma,\tau} = \frac{1+i}{2\sqrt{2}}\left(\sinh\frac{\sigma+\tau}{\sqrt{2}} + i\;\sinh\frac{\sigma-\tau}{\sqrt{2}}\right). \tag{5.4}$$

Eq. (5.2) fully describes the evolution of our adapted frame $\{e_N\}$ along the surface in terms of an initial choice at some starting point. The freedom in this initial choice is related to

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isometry transformations of the surface as a whole. Since $Y(\sigma,\tau)$ is our first vector in the frame, we can read off the coordinates of the surface vector with respect to the $\mathbb{R}^{2,n-1}$ basis $\{e_N(0,0)\}$ from the first row of the matrix $\mathcal{M}$. There is however still one subtlety, due to the fact that the second and third vector of our frame are not normalized and not orthogonal to each other.[3] Orthonormal combinations of ∂Y and $\bar\partial Y$ are $\frac{1}{\sqrt{2}}e^{-\alpha/2}(\partial Y+\bar\partial Y)$ and $\frac{-i}{\sqrt{2}}e^{-\alpha/2}(\partial Y-\bar\partial Y)$ (in these combinations a sign ambiguity, again related to a sign reversal of an embedding coordinate has been fixed). Therefore, to get the coordinates of Y with respect to an orthonormal basis in $\mathbb{R}^{2,n-1}$, one has to take $1/\sqrt{2}$ times the sum and $-i/\sqrt{2}$ times the difference of the second and third entry of the first row of $\mathcal{M}$. A last point to remember is that the two timelike vectors in our frame sit at position 1 and 4. Taking all this into account we get

$$\begin{aligned} Y^0 &= \cosh\frac{\sigma}{\sqrt{2}}\,\cosh\frac{\tau}{\sqrt{2}}\,, & Y^{0'} &= \sinh\frac{\sigma}{\sqrt{2}}\,\sinh\frac{\tau}{\sqrt{2}}\,,\\ Y^1 &= \sinh\frac{\sigma}{\sqrt{2}}\,\cosh\frac{\tau}{\sqrt{2}}\,, & Y^2 &= \cosh\frac{\sigma}{\sqrt{2}}\,\sinh\frac{\tau}{\sqrt{2}}\,. \end{aligned} \tag{5.5}$$

which is the solution used in [1, 2] for the four-point amplitude.

We now turn to the search for flat spacelike minimal surfaces in AdS_5. Then from (4.6) and (4.16) we conclude that there is no such surface of type spacelike I or spacelike III. In case spacelike II, due to (4.11), flatness implies $\cos\beta = -e^{2\alpha}$. As long as $\sin\beta \neq 0$ this gives after differentiation

$$\partial\bar\partial\beta = \frac{4e^{2\alpha}}{\sin\beta}\left(1-\frac{\cos\beta\; e^{2\alpha}}{\sin^2\beta}\right)\partial\alpha\bar\partial\alpha\,. \tag{5.6}$$

Inserting it into (4.12) one arrives at the condition

$$4e^{2\alpha}\,\partial\alpha\bar\partial\alpha + \big(e^{-\alpha}+\rho\bar\rho\big)\big(1-e^{4\alpha}\big)^2 = 0\,, \tag{5.7}$$

which, due to $\rho\bar\rho \geq 0$, cannot be fulfilled.[4]

Therefore, the only remaining possibility is $\sin\beta = 0$, i.e. $\cos\beta = -1$ (the option $\cos\beta = 1$ is excluded by (4.11)). For ρ, $\bar\rho$ eqs. (4.13), (4.14) degenerate to $\partial\bar\rho + \bar\partial\rho = 0$. The matrices A_N^K and $\bar A_N^K$ for eq. (2.8) are then

$$A_N^K = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&-i&0&0\\ 1&0&0&0&0&0\\ 0&0&-i&0&0&0\\ 0&0&0&0&0&i\rho\\ 0&0&0&0&-i\rho&0 \end{pmatrix}, \qquad \bar A_N^K = \begin{pmatrix} 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&i&0&0\\ 0&i&0&0&0&0\\ 0&0&0&0&0&-i\bar\rho\\ 0&0&0&0&i\bar\rho&0 \end{pmatrix}. \tag{5.8}$$

Both matrices are block diagonal. This property will be conserved under exponentiation. As a consequence, the new degrees of freedom relative to the AdS_3 case, encoded in the

[3] For a fully orthonormal choice of the e_N the matrix $\mathcal{M}$ would be $\in \mathrm{SO}(2,n-1)$.

[4] As mentioned already, for timelike surfaces $\rho\bar\rho$ can have both signs, thus allowing more options.

lower right blocks with ρ and $\bar{\rho}$, do not influence the first row of the six-dimensional analog of (5.3).

One can make an even stronger statement on ρ and $\bar{\rho}$. Via a gauge transformation (3.3), acting only in the space spanned by B_5 and B_6, one can achieve $\rho = \bar{\rho} = 0$. This can be seen in two ways. Firstly, with $\partial\bar{\rho} + \bar{\partial}\rho = 0$ one finds zero field strength components related to the lower right corner of (5.8). Secondly going back to (4.3) one finds that, as soon as either $\sin\frac{\beta}{2}$ or $\cos\frac{\beta}{2}$ are zero, u and $\bar{u}$ are parallel. We are just interested in $\cos\beta = -1$ i.e. $\cos\frac{\beta}{2} = 0$. Then eq. (2.14) leads to the vanishing of all components of the field strength tensor already from the very beginning.

Altogether this proves that all flat spacelike minimal surfaces in AdS_5 are realized in a subspace AdS_3, trivially extended into AdS_5, and are of type (5.5).

This statement can be extended in a straightforward manner to $AdS_n, n > 5$. Let us sketch the set of equations one gets instead of (4.5) - (4.18). The Gauß equations (4.6), (4.11) and (4.16) remain unchanged, which again excludes flat minimal surfaces of type spacelike I and III. For the remaining case spacelike II, eq. (4.10) is generalized to ${A_4}^5 = \frac{i}{2}\partial\beta$, ${A_4}^b = \rho^b\cos\frac{\beta}{2}$, ${A_5}^b = i\rho^b\sin\frac{\beta}{2}$, $\quad b = 6, \dots, n+1$. There arise no constraints on ${A_a}^b$ if both $a, b \geq 6$. In eq. (4.12) one has to make the replacement $\rho\bar{\rho} \mapsto \sum_{b=6}^{n+1}\rho^b\bar{\rho}^b$ and in (4.13), (4.14) $\partial\bar{\rho} \mapsto \partial\bar{\rho}_a - {A_a}^b\bar{\rho}_b$. Then the flatness condition necessarily leeds to $\cos\beta = -1$ and a block diagonal structure for A_N^K, $\bar{A}_N^K$ with the (4×4) upper left block of AdS_3 structure and a $(n-3)\times(n-3)$ lower right block.

6 Flat timelike minimal surfaces

The flatness condition implies $\bar{\partial}\partial\alpha = 0$, as above. Together with the sinh-Gordon equation $\bar{\partial}\partial\alpha - 2\sinh\alpha = 0$ in AdS_3, this allows only the vanishing solution $\alpha = 0$, which leads to the rigid infinite rotating string of [4].

In AdS_4 one has two equations (3.9). One solution is $\alpha = 0$, $\cos\beta = -1$. It obviously corresponds to the AdS_3 case extended to AdS_4 trivially. For $\alpha \neq 0$, similarly to the spacelike case, one finds

$$(1 - e^{4\alpha})^2 = -4e^{3\alpha}\partial\alpha\,\bar{\partial}\alpha\ . \tag{6.1}$$

Since for flat surfaces α has a chiral decomposition $\alpha = \phi(z) + \bar{\phi}(\bar{z})$, the r.h.s of eq. (6.1) is given as a product of chiral and antichiral fields. Calculating $\partial\bar{\partial}$ of the logarithm, the r.h.s. is always zero, while the l.h.s. vanishes only for constant ϕ or $\bar{\phi}$. Altogether (6.1) has no solution rather than $\alpha = 0$.

But starting from AdS_5 one can find more flat solutions. An explicit example is the double spin solution of ref. [8]. We follow the scheme of the previous section. The timelike analogs of eqs. (4.11)–(4.12), as mentioned above, are the same. The equation similar to (5.7) provides

$$\rho\,\bar{\rho} = -\frac{4e^{2\alpha}\,\partial\alpha\,\bar{\partial}\alpha}{(1 - e^{4\alpha})^2} - e^{-\alpha}\ . \tag{6.2}$$

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Instead of (4.13)–(4.14) one gets

$$(\bar\rho\partial\beta + \rho\bar\partial\beta)\ \sin\frac{\beta}{2} - (\partial\bar\rho + \bar\partial\rho)\ \cos\frac{\beta}{2} = 0\,, \tag{6.3}$$

$$(\bar\rho\partial\beta - \rho\bar\partial\beta)\ \cos\frac{\beta}{2} + (\partial\bar\rho - \bar\partial\rho)\ \sin\frac{\beta}{2} = 0\ . \tag{6.4}$$

The crucial point is that $\rho\bar\rho$ can have both signs, while it is positive semidefinite for spacelike surfaces.

Nontrivial flat solutions imply $\cos\beta \neq \pm 1$, i.e. $\cos\frac{\beta}{2} \neq 0$ and $\sin\frac{\beta}{2} \neq 0$, that allow to simplify (6.3)–(6.4) in the form

$$\sin\beta\,\partial\bar\rho + \bar\rho\cos\beta\,\partial\beta = \rho\bar\partial\beta\,, \qquad \sin\beta\,\bar\partial\rho + \rho\cos\beta\,\bar\partial\beta = \bar\rho\,\partial\beta\ . \tag{6.5}$$

Due to $\cos\beta = -e^{2\alpha}$, eqs. (6.2) and (6.5) yield

$$\partial\rho = A\,\rho + B\,\rho^3\,, \qquad \bar\partial\rho = C\,\rho + \frac{D}{\rho}\,, \tag{6.6}$$

where the functions A, B, C and D are expressed through $\phi(z)$, $\bar\phi(\bar z)$. Then the consistency condition for (6.6) provides an algebraic (quadratic in ρ^2) equation for ρ. Thus, the chiral and anti-chiral free fields $\phi(z)$ and $\bar\phi(\bar z)$ ($\alpha = \phi + \bar\phi$) parameterize all flat timelike minimal surfaces in AdS_5.

7 Characterization by invariants of minimal surfaces in AdS_n, $n \geq 4$

While the distinction between timelike and spacelike surfaces has a clear geometrical and physical meaning, the various cases in section 4 appeared on a rather technical level using conformal coordinates. To find a characterization, which is both diffeomorphism invariant as well as invariant with respect to local isometry transformations in the normal space, we start with defining as $F = F_{z\bar z}$ the field strength related to $A = A_z$ and $\bar A = A_{\bar z}$, i.e. the r.h.s of eq. (2.14). Next we introduce for $n \geq 4$ the invariant torsion quantity

$$T = \frac{1}{8\,|\det g|}\ \epsilon^{\alpha\beta}\ \epsilon^{\mu\nu}\ \mathrm{tr}(F_{\alpha\beta}F_{\mu\nu})\ . \tag{7.1}$$

Evaluating in conformal coordinates and using eq. (2.14), T becomes

$$T = \frac{1}{2}\ e^{-2\alpha}\ \mathrm{tr}\ F^2 = e^{-4\alpha}\big((\bar u_a u^a)^2 - (\bar u_a \bar u^a)(u_b u^b)\big)\ . \tag{7.2}$$

Due to (2.6) one has $T \leq 0$ for timelike surfaces, while T can have both signs for spacelike surfaces. Furthermore, for timelike surfaces $T = 0 \ \Rightarrow \forall F_a{}^b = 0$. In contrast, in the spacelike case such a conclusion cannot be drawn.

Resolving with respect to $\bar u_a u^a$, putting into the Gauß equation (2.12) and using (2.15), we get with $C = (\bar u_a \bar u^a)(u_b u^b)$

$$R + 2\ \pm\ 2\ e^{-2\alpha}\ \sqrt{C + e^{4\alpha}\ T} = 0\ . \tag{7.3}$$

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Exceptional cases. All exceptional cases, discussed in the previous sections, can be summarized by $C = 0$. Then from (7.2) $T \geq 0$. For timelike surfaces this necessarily means $T = 0$, hence $R + 2 = 0$. The surface is then an $AdS_2 \subset AdS_n$. For the spacelike case the option $T = 0$ gives a surface isometrically to $\mathbb{H}^2$, and for $T > 0$ one can even fix the sign ambiguity coming from (7.3) and gets $R + 2 + 2\,T^{1/2} = 0$.

Non-exceptional cases. Here the choice of coordinates on the surface can be fixed completely such that $C = 1$. Contrary to the exceptional cases, α no longer drops out of (7.3), and one can express α in terms of invariant quantities

$$e^{-4\alpha} = \frac{(R+2)^2}{4} - T \ . \tag{7.4}$$

Altogether, now a nice picture emerges. First of all, as a spin off, we have proven that for *all* minimal surfaces in $AdS_n, \ n \geq 4$

$$\frac{(R+2)^2}{4} - T \ \geq 0 \ . \tag{7.5}$$

This inequality is saturated by the exceptional cases.

For non-exceptional timelike minimal surfaces one has $(R+2)^2 - 4T > 0$, which due to $T \leq 0$ induces no further subdivision.

For non-exceptional spacelike minimal surfaces one gets

$$\begin{aligned} \text{case I}: &\quad 0 \leq T < \frac{(R+2)^2}{4} \,, \\ \text{case II}: &\quad T \leq 0 \,, \\ \text{case III}: &\quad T = 0 \,, \qquad \text{not all} \ \ F_a{}^b = 0 \ . \end{aligned} \tag{7.6}$$

Note that if $T = 0$ in case I or II it results in $F_a{}^b = 0$, as in the timelike case.

8 Conclusions

Along the lines of refs. [4, 11] we have analyzed both timelike and spacelike minimal surfaces in AdS_n. We went beyond these works in two aspects. One concerns the derivation of the differential equations for the reduced system for $n \geq 5$ and the other concerns the parallel treatment of both timelike and spacelike surfaces. In this analysis we pointed out crucial differences in the respective equations. For spacelike minimal surfaces in $AdS_n, \ n \geq 5$ one finds three types of surfaces which differ among themselves in the form of their reduced equations, too.

Based on our analysis, we proved that there are no flat spacelike minimal surfaces in AdS_n, beyond those embedded in an $AdS_3 \subset AdS_n$ (where AdS_3 is totally geodesic in AdS_n) and used for the tetragon case of the Alday-Maldacena conjecture. Furthermore, a parameterization of all flat timelike surfaces in AdS_5 by two free chiral fields has been done.

The considerations are performed in a certain patch of the surface. But since the resulting differential equations yield the globally well defined four cusp solution, the statement can be made concerning surfaces as a whole.

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We stressed that there exist flat timelike minimal surfaces in AdS_5, which cannot be embedded in an AdS_3 subspace [8]. The fact that their double Wick rotation does not yield a spacelike surface in AdS_5 is no accident and finds its deeper explanation in the theorem just stated.

The subdivision for the description of spacelike minimal surfaces, first introduced in the discussion based on conformal coordinates, finds a characterization in terms of the scalar curvature R and a quadratic torsion invariant T. We also derived a universal inequality involving R, T.

There remain a lot of open problems. First of all no progress towards minimal surfaces with higher polygonal boundaries has been achieved.

In the application to the dynamics of open or closed strings the issue of boundary conditions inside AdS becomes relevant and restricts to some extent the allowed conformal transformations on the surface as a whole.

In addition, our analysis generated various other questions already before it comes to the issue of boundary conditions. The reduction of the system for generic AdS_n unfolds interesting structures relevant to the most convenient choice of parameterizing functions and gauge fixing. One can also apply a gauge invariant description using group valued fields instead of connections $(A, \bar{A})$. This approach relates the AdS string dynamics to gauged WZW models [14], similarly to the $AdS \times S$ case [12]. Work in this direction is in progress.

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References

[1] L.F. Alday and J.M. Maldacena, *Gluon scattering amplitudes at strong coupling*, *JHEP* **06** (2007) 064 [arXiv:0705.0303] [SPIRES].

[2] L.F. Alday and J. Maldacena, *Comments on gluon scattering amplitudes via AdS/CFT*, *JHEP* **11** (2007) 068 [arXiv:0710.1060] [SPIRES].

[3] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D 72** (2005) 085001 [hep-th/0505205] [SPIRES].

[4] A. Jevicki, K. Jin, C. Kalousios and A. Volovich, *Generating AdS string solutions*, *JHEP* **03** (2008) 032 [arXiv:0712.1193] [SPIRES].

[5] D. Astefanesei, S. Dobashi, K. Ito and H. Nastase, *Comments on gluon 6-point scattering amplitudes in $N = 4$ SYM at strong coupling*, *JHEP* **12** (2007) 077 [arXiv:0710.1684] [SPIRES];
S. Dobashi, K. Ito and K. Iwasaki, *A numerical study of gluon scattering amplitudes in $N = 4$ super Yang-Mills theory at strong coupling*, *JHEP* **07** (2008) 088 [arXiv:0805.3594] [SPIRES];

S. Dobashi and K. Ito, *Discretized minimal surface and the BDS conjecture in $N = 4$ super Yang-Mills theory at strong coupling*, arXiv:0901.3046 [SPIRES].

[6] A. Mironov, A. Morozov and T.N. Tomaras, *On n-point amplitudes in $N = 4$ SYM*, JHEP **11** (2007) 021 [arXiv:0708.1625] [SPIRES]; *Some properties of the Alday-Maldacena minimum*, Phys. Lett. **B 659** (2008) 723 [arXiv:0711.0192] [SPIRES].

[7] C.M. Sommerfield and C.B. Thorn, *Classical worldsheets for string scattering on flat and AdS spacetime*, Phys. Rev. **D 78** (2008) 046005 [arXiv:0805.0388] [SPIRES].

[8] S. Frolov and A.A. Tseytlin, *Multi-spin string solutions in $AdS_5 \times S^5$*, Nucl. Phys. **B 668** (2003) 77 [hep-th/0304255] [SPIRES].

[9] B.M. Barbashov and V.V. Nesterenko, *Relativistic string model in a space-time of a constant curvature*, Commun. Math. Phys. **78** (1981) 499 [SPIRES].

[10] K. Pohlmeyer, *Integrable Hamiltonian systems and interactions through quadratic constraints*, Commun. Math. Phys. **46** (1976) 207 [SPIRES].

[11] H.J. De Vega and N.G. Sanchez, *Exact integrability of strings in D-Dimensional de Sitter space-time*, Phys. Rev. **D 47** (1993) 3394 [SPIRES].

[12] M. Grigoriev and A.A. Tseytlin, *On reduced models for superstrings on $AdS_n \times S^n$*, Int. J. Mod. Phys. **A 23** (2008) 2107 [arXiv:0806.2623] [SPIRES].

[13] K. Pohlmeyer and K.-H. Rehren, *Reduction of the two-dimensional $O(n)$ nonlinear σ-model*, J. Math. Phys. **20** (1979) 2628 [SPIRES].

[14] I. Bakas, Q.-H. Park and H.-J. Shin, *Lagrangian formulation of symmetric space sine-Gordon models*, Phys. Lett. **B 372** (1996) 45 [hep-th/9512030] [SPIRES].

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